North Carolina Math Olympiad

NCMO Staff

January 2025

Welcome to the North Carolina Math Olympiad! You will have 3 hours to solve 5 problems. Each problem will be graded out of 7 points based on the completeness, clarity, and correctness of your solution.

The only tools permitted are writing utensils, erasers, and straightedge and compass if you have one (optional). For each solution page you wish to submit, write your ID, the problem number, and the page number for that solution in the top right corner:

Contestant ID 999
NCJMO Problem 5
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If you have no experience with proofs, simply write your answer as if you were explaining it to a friend: concisely, clearly, yet with no holes. Even if you don't think you have fully solved a problem, write down and submit whatever progress you have.

This will be a very challenging test, and you should be proud to solve even one problem. Try reading all of the problems and thinking about each one at least briefly, even if you decide to focus on one or two. If you ever you feel tired and need a break, feel free to leave your desk quietly and eat our provided snacks or stretch as needed (within the proctor's view). Good luck and have fun!

NCMO 1. A collection of n positive integers, where repeats are allowed, adds to 500. They can be split into 20 groups each adding to 25, and can also be split into 25 groups each adding to 20. (A group is allowed to contain any amount of integers, even just one integer.) What is the least possible value of n?

If your answer is (for example) 2025, you must (a) describe a collection of 2025 positive integers that works, and (b) explain why the collection cannot be any smaller.

NCMO 2. In pentagon $ABCDE$, the altitudes of triangle ABE meet at point H. Suppose that $BCDE$ is a rectangle, and that B, C, D, E, and H lie on a single circle. Prove that triangles ABE and HCD are congruent.

NCMO 3. Let $\mathcal{S}$ be a set of points in the plane such that for each subset $\mathcal{T}$ of $\mathcal{S}$, there exists a convex 2025-gon which contains all of the points in $\mathcal{T}$ and none of the rest of the points in $\mathcal{S}$ but not $\mathcal{T}$. What is the greatest possible number of points in $\mathcal{S}$?

If your answer is (for example) 2025, you must (a) describe a set of 2025 points that works, and (b) explain why the set cannot have any more points.

NCMO 4. Let P be a polynomial. Suppose that there exists a rational constant q such that $P(m) = q^n$ for infinitely many integers (m, n). Prove that $P(x) = c \cdot Q(x)^k$ for some integer constants c and k and irreducible polynomial Q with rational coefficients.

(Here, a polynomial is *irreducible* if it cannot be factored into the product of two non-constant polynomials with rational coefficients.)

NCMO 5. Let x be a real number. Suppose that there exist integers $a_0, a_1, \ldots, a_n$, not all zero, such that

$$\sum_{k=0}^{n} a_k \cos(kx) = \sum_{k=0}^{n} a_k \sin(kx) = 0.$$

Characterize all possible values of $\cos x$.

If your answer is (for example) -1 and 1 only, you must (a) explain why $\cos x = \pm 1$ are attainable, and (b) explain why no other values of $\cos x$ are attainable.