NC(J)MO 2025 Results and Solutions

NCMO Staff

January 2025

Acknowledgments

We deeply appreciate Ms. Kira Viola for her invaluable support in hosting the Charlotte site at Myers Park HS. Also, we would like to thank the following individuals for helping us promote the Olympiad: Dr. Harold Reiter, Allen Liu from Enloe Magnet HS, Siddharth Senthil from Lake Norman Charter, Jett Mu from NCSSM-Durham, Lingfei Tang from Cary Academy, and Ms. Alex Hanner from AoPS Morrisville.

NCMO Winners

Name	Grade	School	P1	P2	P3	P4	P5	Total
Watson Houck	9	Myers Park	7	7	7	1	0	22
Tatiana Medved	12	NCSSM-Durham	7	7	1	-	7	22
Lucas Li	8	Lake Norman	7	7	7	-	-	21
Jett Mu	12	NCSSM-Durham	7	7	7	-	0	21
Spencer Thomas	12	NCSSM-Durham	7	7	7	0	0	21

NCJMO Winners

Name	Grade	School	P1	P2	P3	P4	P5	Total
Jimmy Chen	11	NCSSM-Durham	7	3	6	7	1	24
Karthi Palanichamy	7	Lake Norman	7	3	7	0	0	17
Jake Baugh	12	Myers Park	7	3	6	-	-	16
Sachetan Bellanki	8	Carnage Magnet	6	7	-	2	-	15
Siddhartha Bajpai	10	Ardrey Kell	7	0	7	0	0	14

Special Award

The **Brilliance Award** goes to Oliver Lippard, a homeschooled senior, for his extraordinary progress on Problems 4 and 5 of the NCMO.

NCJMO P1, proposed by Jason Lee

Problem. Cerena, Faith, Edna, and Veronica each have a cube. Aarnő knows that the side lengths of each of their cubes are distinct integers greater than 1, and he is trying to guess their exact values. Each girl fully paints the surface of her cube in Carolina blue before splitting the entire cube into $1 \times 1 \times 1$ cubes. Then,

- Cerena reveals how many of her $1 \times 1 \times 1$ cubes have exactly 0 blue faces.
- Faith reveals how many of her $1 \times 1 \times 1$ cubes have exactly 1 blue faces.
- Edna reveals how many of her $1 \times 1 \times 1$ cubes have exactly 2 blue faces.
- Veronica reveals how many of her $1 \times 1 \times 1$ cubes have exactly 3 blue faces.

Whose side lengths can Aarnő deduce from these statements?

Solution. The answer is Cerena, Faith, and Edna. Let's replace the wordy phrase "$1 \times 1 \times 1$ cube" with the word "cubie," to save our breath. Also, it may help to grab a *Rubik's cube* (preferably $4 \times 4 \times 4$ or bigger) to follow along with the solution.

- Call c the side length of Cerena's cube. Her cubies with 0 blue faces are those of the central $(c-2) \times (c-2) \times (c-2)$ cube that is hidden from the outside eye. Hence, Cerena reveals the number $(c-2)^3$, and Aarnő can set this expression equal to the number he hears to solve for c.
- Call f the side length of Faith's cube. Her cubies with 1 blue face are those of the middle $(f-2) \times (f-2)$ squares on each of the 6 faces of the big cube. So Faith reveals the number $6(f-2)^2$, and Aarnő can backsolve for f just like before.
- Call e the side length of Edna's cube. Her cubies with 2 blue faces are those of the middle $e-2$ "rods" on the 12 edges of the big cube. Thus, Edna reveals $12(e-2)$ and Aarnő can solve for e.
- No matter the side length of Veronica's cube, she will always have 8 cubies with 3 blue faces, one at each corner/vertex. Aarnő can't tell what her side length is.

In summary, Aarnő can deduce everyone's side lengths except Veronica's. □

Remarks. Did you notice that **CE**rena, **FA**ith, **ED**na, and **VER**onica reveal how many **CE**nter, **FA**ce, **ED**ge, and **VER**tex (corner) cubies they have? Also, **AARNŐ** is a portmanteau of **AAR**on, one of our staff members, and E**RNŐ**, the first name of the Rubik's cube's founder and namesake!

Interestingly, when we add up the number of center, face, edge, and corner cubies for an $n \times n \times n$ cube, we get a *binomial expansion*:

$$(n-2)^3 + 6(n-2)^2 + 12(n-2) + 8 = [(n-2)+2]^3 = n^3.$$

See if you can find out why! (Hint: *rearrange* the cubies.)

Rubric.

- **0, 1, 3, 5, and 7 pts** for respectively answering and explaining for 0, 1, 2, 3, and 4 girls correctly
- **-1 pt** *overall* for arguing that the side length of the cube *increases with* Cerena's, Faith's, and Edna's quantities without explaining why it is *uniquely determinable* by those quantities
- **-1 pt** *for each* answer lacking an accompanying explanation

NCJMO P2 / NCMO P1, proposed by Aaron Wang

Problem. A collection of n positive numbers, where repeats are allowed, adds to 500. They can be split into 20 groups each adding to 25, and can also be split into 25 groups each adding to 20. (A group is allowed to contain any amount of numbers, even just one number.) What is the least possible value of n?

Solution. The answer is $n = 40$.

Construction (showing that $n = 40$ is attainable). We present two constructions, both of which fit the bill equally well:

1. Take 20 20's and 20 5's. We can split them into 20 groups of the form $\{20, 5\}$; and we can split them into 20 groups of $\{20\}$ and 5 groups of $\{5, 5, 5, 5\}$.
2. Take the number line from 0 to 500, and connect the endpoints to form a circle. If we cut the circle at multiples of 20 and 25, the lengths of the resulting arcs will be our collection. As there are $\frac{500}{20} = 25$ multiples of 20, $\frac{500}{25} = 20$ multiples of 25, and $\frac{500}{\operatorname{lcm}(20,25)} = 5$ multiples of both, there are $20 + 25 - 5 = 40$ arcs.

Bound (proving that $n = 40$ is smallest). Each integer must be at most 20 in order to fit into some group adding to 20. Hence, each of the 20 groups adding to 25 must have at least 2 numbers, so $n \geq 2 \cdot 20 = 40$. $\square$

Remarks. This problem is deceptively simple, and part of why it's subtly difficult (especially to more experienced contestants) owes to the fact that it isn't easily *generalizable*. In particular, the 1$^{\text{st}}$ construction doesn't easily generalize to (a, b) other than $(20, 25)$, and the 2$^{\text{nd}}$ construction and bound give $n = a + b - \gcd(a, b)$ and $n \geq 2\min(a, b)$, respectively. These values match only when $|a - b| = \gcd(a, b)$.

Funnily enough, Aaron wrote this back in 2024, so the problem statement had the values $(a, b) = (20, 24)$. In that case, we still have $|a - b| = \gcd(a, b)$ and even the same final answer of $n = 40$!

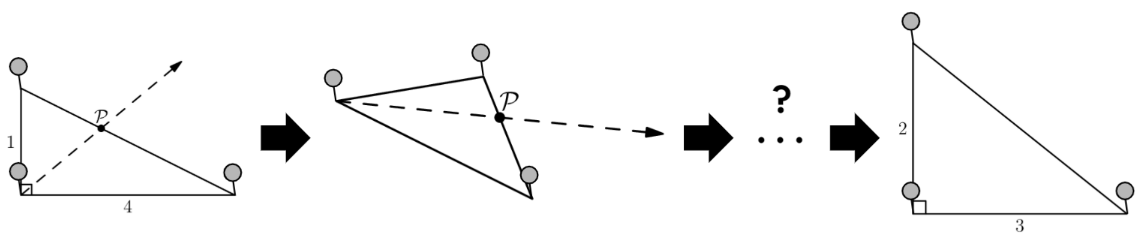
Rubric:

- **0 pts** for answering $n = 1$ (misinterprets the problem), $n = \text{lcm}(20, 25)$ (assumes all numbers are equal), or other significantly erroneous answers
- **+1 pt** for observing that every number is at most 20
- **+2 pts** for answering $n = 40$ alongside a valid collection of 40 numbers (with or without an explanation of why it's valid)
- **7 pts** for correct answer, construction, and $n \geq 2 \cdot 20$ proof of bound

In particular, note that any proof along the lines of "to minimize n, we need to maximize the number of 20's" earns at most 3 points. Indeed, there exist equally good constructions with 40 numbers that don't have the maximum number of 20's.

NCJMO P3, proposed by Jason Lee

Problem. Alan has three pins that form a right triangle with legs 1 and 4 at first. Every move, he can pick any one of the pins, pick any new point $\mathcal{P}$ on the opposite side, and move the pin to its *reflection* across $\mathcal{P}$. After a series of moves, can the pins eventually form a right triangle with legs 2 and 3? Explain your answer.



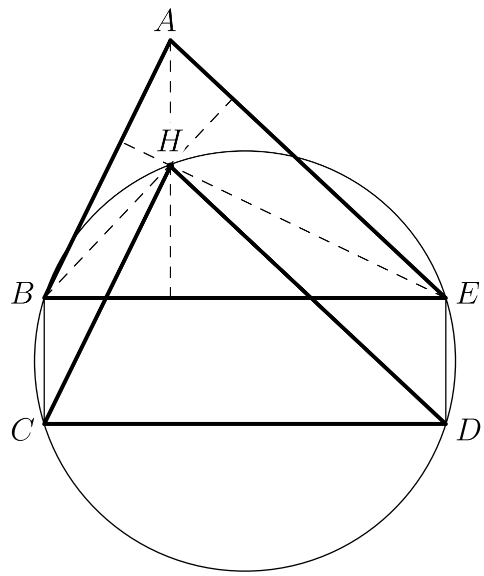
Solution. The answer is no. We claim that the *area* of the triangle is *invariant*, i.e., it never changes throughout the process. Why? Whenever we reflect one of the pins across some point $\mathcal{P}$ on the opposite side, that opposite side is shared by the old and new triangles, and the height from that pin to the opposite is the same in both triangles. Hence, by the formula $\frac{\text{base}\times\text{height}}{2}$, the area stays the same. However, since the area of the starting triangle is $\frac{1\times 4}{2} = 2$, while the area of the last triangle is $\frac{2\times 3}{2} = 3$, there is no valid sequence of reflections that turns the 1×4 triangle into the 2×3 triangle. $\square$

Rubric.

- **0 pts** for guessing the correct answer without any relevant proof
- **1 pt** for claiming some *incorrect* invariant such as perimeter
- **−1 pt** for finding the area invariant but accompanying it with false claims (e.g. perimeter invariant as well)

NCJMO P4 / NCMO P2, proposed by Alan Cheng

Problem. In pentagon $ABCDE$, the altitudes of triangle ABE meet at point H. Suppose that $BCDE$ is a rectangle, and that B, C, D, E, and H lie on a single circle. Prove that triangles ABE and HCD are congruent.



Solution 1 (Staff).

- EH is perpendicular to AB, by definition, and to CH, since CE is a diameter. Hence, AB and CH are parallel. Also, AH and BC are both perpendicular to BE, and so they are parallel to each other. Combining these, we see that $ABCH$ is a parallelogram with $AB = CH$ and $AH = BC$.

- Just like before, $AE \perp BH \perp DH$ and $AH \perp BE \perp DE$, and so $AEDH$ is a parallelogram with $AE = DH$ and $AH = DE$.

- Since $BCDE$ is a rectangle, it is a parallelogram with $BE = CD$ and $BC = DE$.

Now we can either finish with SSS ($AB = CH$, $AE = DH$, $BE = CD$), or by noting that $\triangle ABE$ and $\triangle HCD$ are translations of each other by the common length $AH = BC = DE$ in the direction perpendicular to BE. $\square$

Solution 2 (Jack Whitney-Epstein & Author). Since A is the orthocenter of $\triangle HBE$, it is well-known that its reflection A' over BE lies on the circle. Then $\triangle ABE$ is the reflection of $\triangle A'BE$ across BE, and $\triangle A'BE$ is the reflection of $\triangle HCD$ across the horizontal midline of $BCDE$, and so all three triangles are congruent. $\square$

Solution 3 (Jiahe Liu). Set the circle to be the unit circle. Since A is the orthocenter of $\triangle HBE$, $\vec{A} = \vec{H} + \vec{B} + \vec{E}$. Then the triangle formed by $\vec{A} = \vec{H} + \vec{B} + \vec{E}$, $\vec{B}$, and $\vec{E}$ is simply a translation by $\vec{B} + \vec{E}$ of the triangle formed by $\vec{H}$, $\vec{C} = -\vec{E}$, and $\vec{D} = -\vec{B}$. $\square$

Rubric.

- **+1 pt** for at least one of the following items:
  - proving $BE = CD$, claiming $AB = CH$ and $AE = DH$ without proof, and finishing by SSS congruence
  - proving $BE = CD$, claiming $\angle ABE = \angle HCD$ and $\angle AEB = \angle HDC$ without proof, and finishing by ASA congruence
  - proving $AH \parallel BC \parallel DE$, claiming $AH = BC = DE$ without proof, and finishing by translation
- **+1 pt** for angle chasing along the circle to find $\angle BHD = 90^\circ$ or some nontrivial equal pair of angles
- **+3 pts** for proving $AB \parallel CH$, $AB = CH$, or similar, but failing to finish

NCJMO P5, proposed by Grisham Paimagam

Problem. Imagine that everyone in a room is wearing many colored wristbands. Each wristband has a single color, but there are many different colors of wristbands.

A "rainbow" is subset of the room where each person in the room shares a wristband color with at least one member of the rainbow. For example, the subset containing everyone is of course a rainbow, since everyone in the room shares a color with some member of the rainbow (namely, themself).

A "minimal rainbow" is a rainbow where removing any one of its members gives a non-rainbow. Prove that every person in the room is part of at least one minimal rainbow.

Solution 1 (Staff). View the people in the room as the vertices of a graph, with edges connecting people that share at least 1 wristband color. For any vertex v, construct the following subset $\mathcal{S}$ containing it:

1. Start with the subset $\mathcal{S} = \{v\}$.
2. *Add* a vertex that doesn't neighbor any vertex currently in the subset.
3. Repeat until no such vertex exists.

We claim that $\mathcal{S}$ is a minimal rainbow. Indeed, it is a rainbow since, by definition, everyone neighbors an vertex of $\mathcal{S}$ at the end of the process. Further, it is minimal since, again by definition, removing any $v' \in \mathcal{S}$ would mean v' doesn't neighbor any element in $\mathcal{T}$, contradiction. Hence, every vertex v is part of a minimal rainbow. $\square$

Solution 2 (Alexander Wang). Construct the same graph as before, and for any vertex v, construct the following subset $\mathcal{S}$ containing it:

1. Start with the subset $\mathcal{S}$ containing v and all the vertices not neighboring v. This is a rainbow, since everyone in the room either shares a color with v (which is in the rainbow) or is already in the rainbow and shares a color with themself.
2. *Remove* a vertex such that the new $\mathcal{S}$ is still a rainbow.
3. Repeat until no such vertex exists.

Clearly the result of this process is a minimal rainbow, so it suffices to show we can never remove v. But if we ever remove v, v will not share a color with anyone in $\mathcal{S}$, so the new $\mathcal{S}$ could not be a rainbow, contradiction. Thus the final $\mathcal{S}$ is a minimal rainbow containing v. □

Remarks. Here's the author's original formulation of the problem, which is much more succinct and slightly more difficult psychologically (due to the term "union"):

Each element of set $\mathcal{S}$ is colored with multiple colors. A *rainbow* is a subset of $\mathcal{S}$ which has amongst its elements at least 1 color from each element of $\mathcal{S}$. A *minimal rainbow* is a rainbow where removing any single element gives a non-rainbow. Prove that the union of all minimal rainbows is $\mathcal{S}$.

Rubric.

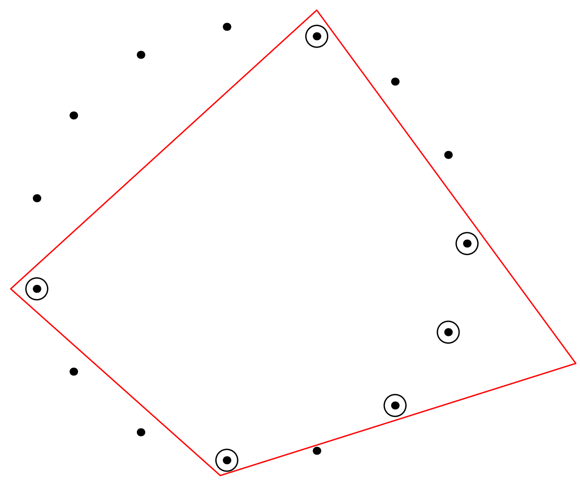
- **0 pts** for trivializing assumptions (e.g. the singletons are minimal rainbows, each person has one wristband)
- **0 pts** for constructing directly with some global property (e.g. each person has a "unique color") without further elaboration
- **+1 pt** for some sort of constructive approach, i.e., starting with some arbitrarily large rainbow containing v and removing vertices

NCMO P3, proposed by Jason Lee

Problem. Let $\mathcal{S}$ be a set of points in the plane such that for each subset $\mathcal{T}$ of $\mathcal{S}$, there exists a convex 2025-gon which contains all of the points in $\mathcal{T}$ and none of the rest of the points in $\mathcal{S}$ but not $\mathcal{T}$. Determine the greatest possible number of points in $\mathcal{S}$.

Solution. The answer is 4051, generally $2m + 1$ for 2025 replaced by m. Clearly we may assert that $\mathcal{S}$ is convex, since otherwise any convex 2025-gon containing $\mathcal{T}$ the convex hull will inevitably contain any points strictly inside the convex hull as well, rendering $\mathcal{T}$ the convex hull unattainable.

The key idea is to consider "clusters" of consecutive vertices that are excluded from $\mathcal{T}$. Observe that each cluster must be separated from the rest of the points by some side of the 2025-gon. Further, no two clusters can be separated from the rest of the points by the same side of the 2025-gon. As long as these conditions hold, we can let these separating lines meet to become the sides of our desired 2025-gon (adding as many negligible sides as are necessary).



When $\mathcal{S}$ is a convex 4051-gon, each subset $\mathcal{T}$ produces at most 2025 clusters, and so each can have its own separating side, giving the desired 2025-gon that only covers $\mathcal{T}$. On the other hand, if $\mathcal{S}$ has at least 4052 vertices, the alternating subset $\mathcal{T}$ has at least 2026 clusters, impossible. $\square$

Remarks. The above problem is classical in the field of *Vapnik–Chervonenkis theory.* A set $\mathcal{S}$ is said to be *shattered* by a collection $\mathcal{C}$ of sets if, for each $\mathcal{T} \subseteq \mathcal{S}$, there exists some $\mathcal{C}_i \in \mathcal{C}$ such that $\mathcal{C}_i \cap \mathcal{S} = \mathcal{T}$. Then the *Vapnik–Chervonenkis dimension* of a collection $\mathcal{C}$ is the cardinality of the largest set it shatters. For example, the above problem shows that the VC dimension of closed convex m-gons in $\mathbb{R}^2$ is $2m + 1$.

The VC dimension has applications in neural networks and statistical neural theory. For example, the collection of half-planes in $\mathbb{R}^2$ can accurately classify ≤ 3 points; and generally, the VC dim. of a classification model reflects its "capacity" or "complexity".

Rubric.

- **+0 pts** for stating the answer is 3, 2025, or some similarly erroneous value
- **+1 pt** for showing that $\mathcal{S}$ can be assumed to be convex
- **+2 pts** for correct answer and construction
- **−0 pts** for no explanation of construction *if* the bound illustrates the cluster idea clearly enough

NCMO P4, proposed by Jason Lee

Problem. Let P be a polynomial. Suppose that there exists a rational constant q such that $P(m) = q^n$ for infinitely many integers (m, n). Prove that $P(x) = c \cdot Q(x)^k$ for some integer constants c and k and irreducible polynomial Q with rational coefficients.

(Here, a polynomial is *irreducible* if it cannot be factored into the product of two non-constant polynomials with rational coefficients.)

Solution. By interpolating through $\deg P + 1$ such (m, q^n), we see that P has rational coefficients $a_0, \ldots, a_d$. Hence, for any prime p,

$$nv_p(q) = v_p(q^n) = v_p(P(m)) \geq \min(v_p(a_i)).$$

As n takes on infinitely many values, it is unbounded in some direction, say positively; sending $n \to +\infty$ forces $v_p(q) \geq 0$. Arguing over all p gives that q is an integer. (If n is negatively unbounded, mapping $(n, q) \mapsto (-n, \frac{1}{q})$ brings us back to the positive case.)

Suppose toward contradiction that $P \equiv c \cdot Q_1 \cdot Q_2$, where Q_1 and Q_2 share no factors in $\mathbb{Z}[x]$. For convenience, redistribute constant factors such that c is the reciprocal of an integer c' and the Q_i's have integer coefficients.

Claim (Bézout's lemma). As m increases, $\gcd(Q_1(m), Q_2(m))$ is bounded above.
Proof. WLOG let $\deg Q_1 \geq \deg Q_2$, and recursively define Q_{i+1} as the polynomial remainder when Q_{i-1} is divided by Q_i. Inductively, Q_i is always a nonzero polynomial combination of Q_1, Q_2, since (Q_i, Q_{i+1}) always share no factors in $\mathbb{Z}[x]$. Now $\deg Q_i$ is decreasing after $i = 2$, and so Q_i is eventually constant at, say, $i = t$. Clearing denominators, we see that for every m, there exist integers $a(m)$ and $b(m)$ such that

$$Q_t(m) = a(m)Q_1(m) + b(m)Q_2(m) = c,$$

and so $\gcd(Q_1(m), Q_2(m))$ always divides c.

Return to the equation in question:

$$P(m) = q^n \iff Q_1(m)Q_2(m) = c'q^n.$$

The factors of c' can be distributed to Q_1 and Q_2 in finitely many ways. Further, for any prime $p \mid q$, the above claim implies that $(v_p(Q_1(m)), v_p(Q_2(m))) = (O(1), O(n))$ in some order. Arguing over all p, we have a finite number of solution families of the form $(Q_1(m), Q_2(m)) = (c_1 d_1^n, c_2 d_2^n)$, where the c_i's and d_i's are constants.

Finally, it remains to show that in each such family, i.e., under a fixed choice of the c_i's and d_i's, there are finitely many (m, n) satisfying $(Q_1(m), Q_2(m)) = (c_1 d_1^n, c_2 d_2^n)$. But

$$\frac{Q_1(m)^{\deg Q_2}}{Q_2(m)^{\deg Q_1}} = \text{SomeConstant} \left(\frac{d_1^{\deg Q_2}}{d_2^{\deg Q_1}} \right)^n.$$

The LHS converges to a nonzero constant, but the RHS converges to 0 or diverges. $\square$

Remarks. See Post #3 in the thread for the backstory.

Seeing as each version of the problem has been a generalized version of the previous, one might wonder whether the current statement is maximally generalized. It turns out that preserving the main ideas of the solution, that is indeed the case:

- The condition $(m, n) \in \mathbb{Z}^2$ cannot be weakened to $\mathbb{Q}^2$, since the latter is dense over $\mathbb{R}^2$. In particular, the contradiction is fundamentally algebraic (convergence) rather than number theoretic (v_p or similar), which breaks down when there are infinitely many solutions in a finite interval.

- The other direction (i.e., $P \equiv cQ^k \implies \exists\ q$) is likely non-elementary, since it deals with equations of the form $m^3+5 = 7^n$ which likely require elliptic firepower.

The core of the proof is the claim, that being the application of Bézout's lemma over the rational polynomials $\mathbb{Z}[x]$. In its generality, it states that for $P, Q \in K[x]$ with K a field, there exist $a, b \in K[x]$ with $aP + bQ = 1$ if and only if P and Q share no roots in any algebraically closed field, commonly $\mathbb{C}$. Extending to any number of polynomials and variables gives Hilbert's Nullstellensatz.

Finally, if you're unconvinced by the interpolation step at the beginning of the proof, you can write it out explicitly: denote $d := \deg P$ and $P = \sum a_i x^i$; and pick distinct $(m_0, n_0), \ldots, (m_d, n_d) \in \mathbb{Z}^2$ satisfying $P(m_i) = q^{n_i}$, which are given to exist. Encoding these values in a matrix,

$$\begin{bmatrix} 1 & m_0 & \ldots & m_0^d \\ 1 & m_1 & \ldots & m_1^d \\ \vdots & \vdots & \ddots & \vdots \\ 1 & m_d & \ldots & m_d^d \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_d \end{bmatrix} = \begin{bmatrix} q^{n_0} \\ q^{n_1} \\ \vdots \\ q^{n_d} \end{bmatrix}.$$

Rewrite as $\mathbf{Ma} = \mathbf{q}$. Since $\det(\mathbf{M})$ is homogeneous in the m_i's, has degree $\binom{d+1}{2}$, and vanishes if $m_i = m_j$, it is precisely $\prod(m_i - m_j)$, which is nonzero by assumption of distinctness. So $\mathbf{M}$ is invertible and $\mathbf{a} = \mathbf{M}^{-1}\mathbf{q}$; but the latter is in $\mathbb{Q}^{d+1}$ as $\mathbb{Q}$ is closed under multiplication and division. Hence $\mathbf{a} \in \mathbb{Q}^{d+1}$ and $P \in \mathbb{Q}[x]$. (Here $\mathbf{M}$ is commonly known as a Vandermonde matrix.)

Rubric.

- **+1 pt** for intuiting the overall line of attack, i.e., if for contradiction $P = Q_1 \cdot Q_2$ for coprime Q_1, Q_2, Q_1 and Q_2 cannot share prime factors infinitely often

- **+1 pt** for rigorizing said notion using Bézout's, a division algorithm, or similar

- **+1 pt** for showing we can assert q, P, etc. are all integers

NCMO P5, proposed by Grisham Paimagam

Problem. Let x be a real number. Suppose that there exist integers $a_0, a_1, \dots, a_n$, not all zero, such that

$$\sum_{k=0}^{n} a_k \cos(kx) = \sum_{k=0}^{n} a_k \sin(kx) = 0.$$

Characterize all possible values of $\cos x$.

Solution 1 (Staff). The answer is all roots in $[-1, 1]$ of integer-coefficient polynomials. Necessity follows since each $\cos kx$ can be expanded as a Chebyshev polynomial in $\cos x$, so the first sum is identically an integer polynomial in $\cos x$ which vanishes.

By adding the first equation to i times the second, Euler's formula gives

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0$$

where $z := e^{ix}$. Hence, sufficiency is equivalent to showing that if $\cos x$ is the root of an integer polynomial, so is z.

Let $P(t)$ be the integer polynomial of which $\cos x$ is a root; clearly, $2\cos x$ is a root of the integer polynomial $Q(t) := 2^{\deg P} P(\frac{t}{2})$. Now, let $r_1, r_2, \dots$ be the roots of Q, and consider the integer polynomial

$$R(t) := t^{\deg Q} Q(t + \tfrac{1}{t}) = a(t^2 - r_1 t + 1)(t^2 - r_2 t + 1) \dots$$

One of the factors on the RHS is $t^2 - 2\cos x + 1$, which is precisely $(t - z)(t - \bar{z})$. Hence z is a root of the integer polynomial R, as desired. $\square$

Solution 2 (Vikram Sarkar et al). The answer is all algebraic numbers in $[-1, 1]$. As before, the given condition is equivalent to z being algebraic. If z is algebraic, we easily have that $\frac{z+\bar{z}}{2} = \cos x$ is algebraic as well. Conversely, if $\cos x$ is algebraic, the *closure* of the nonzero algebraic numbers under elementary operations implies that

$$\cos x + i \cdot \sqrt{1 - \cos^2 x} = z$$

is algebraic as well. Hence our answer is necessary and sufficient. $\square$

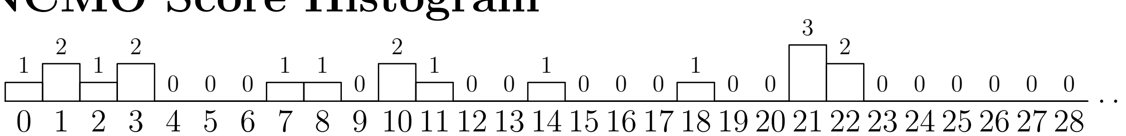
Rubric.

- **+1 pt** for correct answer
- **+1 pt** for proving necessity
- **+1 pt** for considering $z = e^{ix}$
- **−1 pt** for not proving closure of the algebraic numbers

NCMO Problem Statistics

Pts	P1	P2	P3	P4	P5
N/A	0	2	3	11	10
0	2	4	7	5	6
1	1	2	3	1	0
2	1	1	0	1	1
3	4	1	0	0	0
4	0	0	1	0	0
5	0	0	0	0	0
6	0	0	0	0	0
7	10	8	4	0	1
Avg	4.7	3.5	1.9	0.2	0.5
%5+	56	44	22	0	5

NCMO Score Histogram



NCJMO Problem Statistics

Pts	P1	P2	P3	P4	P5
N/A	2	3	8	7	12
0	2	11	19	13	21
1	2	0	3	14	3
2	0	7	0	1	0
3	4	9	0	0	0
4	0	1	0	0	0
5	2	2	0	0	0
6	4	0	3	0	0
7	20	3	3	1	0
Avg	5.2	2.1	1.2	0.6	0.1
%5+	72	14	17	3	0

NCJMO Score Histogram